Cellular automaton traffic flow model between the Fukui-Ishibashi and Nagel-Schreckenberg models

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We propose and study a one-dimensional traffic flow cellular automaton model of high-speed vehicles with the Fukui-Ishibashi-type acceleration for all cars, and the Nagel-Schreckenberg-type (NS) stochastic delay only for cars following the trail of the car ahead. The main difference in the delay scenario between our model and the NS model is that a car with spacing ahead longer than the velocity limit M may not be delayed in our model. By using a car-oriented mean-field theory, we analytically derive fundamental diagrams of the average speed as a function of the car density. Our theoretical results are in excellent agreement with numerical simulations.

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I. INTRODUCTION

Traffic flow cellular automaton (CA) models have attracted much interest recently. Compared with other dynamical approaches, e.g., the fluid dynamical approach, to this class of problems, CA models are conceptually simpler, and can be easily implemented on computers for numerical investigations [1–4].

Two popular one-dimensional (1D) traffic flow models are the Fukui-Ishibashi (FI) model [5] and the Nagel-Schreckenberg (NS) model [6]. An exact car-oriented mean-field (COMF) theory has been developed for the FI model, with an arbitrary limit on the maximum speed $v_{\max}$, car density ρ, and delay probability f [7,8]. However, for the NS model with high-speed vehicles ($v_{\max}>1$) and stochastic delay, to our knowledge there has been no established exact analytical theory up to now [9,10].

The acceleration and stochastic delay rules of the NS model lead to complications in the time evolution of the flow, and hence it is very difficult for exact analytical studies. In order to understand how these rules affect the evolution and the corresponding asymptotic state, we study a 1D traffic flow CA model in which only the cars following the trail of the car ahead may be delayed.

The plan of the present paper is as follows. The definition of the model and the evolution equations for the intercar spacings are given in Sec. II. In Sec. III some observations are made to describe the steady state of the system. In Sec. IV, we present the fundamental diagrams for the low-car-density case with an arbitrary vehicle speed limit, and the high density case with vehicle speeds limited to 1 and 2. Excellent agreements between numerical simulations and theoretical results are shown in Sec. V, together with a discussion of our results in connection to the FI and NS models.

II. MODEL

Let N be the total number of cars on a 1D road of length L. The density of cars is $\rho=N/L$. Let $C_n(t)$ be the number of empty sites in front of the nth car at time t, and $v_n(t)$ be the number of sites that the nth car moves during the time t step. Our model adopts the following acceleration rule [5].

Step 1: $v_n'(t)=\min(C_n(t),M)$. We call the nth car "a car that follows the trail of the car ahead" if $v_n'(t)=C_n(t)$. This means that the nth car may become the neighbor of the car ahead if the car in front stops. Stochastic delay is introduced in such a way that all the cars which follow the trail of their cars ahead have a probability f to move forward one site less than it is allowed by step 1, i.e., we have the following.

Step 2: $v_n(t)=v_n'(t)-1$, with the probability f, if $v_n'(t)=C_n(t)$ and $v_n'(t)>0$.

Step 3: The nth car moves $v_n(t)$ sites ahead.

The number of empty sites in front of the nth car at time $t+1$ can be written as

$$C_n(t+1)=C_n(t)+v_{n+1}(t)-v_n(t). \tag{1}$$

For our model, with a maximum car velocity $v_{\max}=M$ and a stochastic delay probability f, the velocity of the nth car at time step t as a function of the intercar spacing $C_n(t)$ can be written as

$$v_n(t)=F_M[f,C_n(t)] \tag{2}$$

where

$$F_M(f,C)=\begin{cases} M & \text{if } C>M \\ \left.\begin{matrix} C-1 & \text{with probability } f \\ C & \text{with probability } 1-f \end{matrix}\right\} & \text{if } 0<C\leqslant M \\ 0 & \text{if } C=0. \end{cases} \tag{3}$$

III. INTERCAR SPACINGS IN THE STEADY STATES

From Eqs. (1)–(3), we can derive the properties of the intercar spacings in the steady states. Given $C_n(t) \leqslant M+1$, it follows that $C_n(t) - F_M[C_n(t)] \leqslant 1$, and from $F_M[C_{n+1}(t)] \leqslant M$, we obtain $C_n(t+1) = C_n(t) - F_M[C_n(t)] + F_M[C_{n+1}(t)] \leqslant M+1$. Therefore, if an intercar spacing is not larger than $M+1$, it will not be larger than $M+1$ as the system evolves.

Given $C_n(t) \geqslant M+1$, it follows that $F_M[C_n(t)] = M$, and from $F_M[C_{n+1}(t)] \leqslant M$ we have $C_n(t+1) = C_n(t) - F_M[C_n(t)] + F_M[C_{n+1}(t)] \leqslant C_n(t)$. Therefore, intercar spacings which are larger than or equal to $M+1$ will never increase, i.e., if $C_n(t) \geqslant M+1$, then $C_{n+1}(t) \leqslant C_n(t)$.

It is useful to define the long and short intercar spacings via their comparison with the maximum car speed M. An intercar spacing is called a long spacing if it is longer than $M+1$, i.e., $C_n(t) > M+1$. An intercar spacing is called a short spacing if it is shorter than $M+1$, i.e., $C_n(t) < M+1$. Based on the above definitions, we can define the excessive length of a long spacing $L_n(t)$ and the deficient length of a short spacing $S_n(t)$ as

$$L_n(t) = \max[C_n(t) - (M+1), 0]$$

and

$$S_n(t) = \max[(M+1) - C_n(t), 0].$$

It follows that the sum of the excessive lengths of all long spacings $L(t)$ and the sum of deficient lengths of all short spacings $S(t)$ are given, respectively, by

$$L(t) = \sum_n L_n(t), \quad S(t) = \sum_n S_n(t).$$

From these definitions, it can be proven readily that

$$L(t) - S(t) = \sum_n [C_n(t) - (M+1)] = L - (M+2) = \text{const.} \tag{4}$$

From these properties of the intercar spacings, we have $L_n(t+1) \leqslant L_n(t)$. Hence

$$L(t+1) \leqslant L(t). \tag{5}$$

From Eqs. (4) and (5), we have $S(t+1) \leqslant S(t)$. Therefore, L and S will never increase as the system evolves. If one of the L_n decreases, then L and S will have to decrease.

Next we look into the question of whether long and short spacings may coexist in an asymptotic steady state. Let $N_i(t)$ be the number of intercar spacings with length i at time t. The probability of finding such a spacing at time t is $P_i(t) = N_i(t)/N$. Hereafter, $P_i(t)$ is denoted by P_i for simplicity, except if specified otherwise. Suppose that long and short spacings coexist. Consider a long spacing; if the car ahead moves forward by $m-1$ sites, then the spacing will decrease by 1. The probability for this to occur is $(1-f)P_{M-1} + fP_M$. For the same reason, the probability for the spacing to be shortened by 2 is $(1-f)P_{M-2} + fP_{M-1}$, and the probability for the spacing to be shortened by 3 is $(1-f)P_{M-3} + fP_{M-2}$, and so on. The probability for the spacing to be shortened by $M-1$ is $(1-f)P_1 + fP_2$, and the probability for the spacing to be shortened by M is $P_0 + fP_1$. On average, a long spacing will be shortened by

$$MP_0 + \sum_{i=1}^{M-1} [(1-f)(M-i) + f(M-i+1)]P_i + fP_M$$
$$= MP_0 + \sum_{i=1}^{M-1} (M-i+f)P_i + fP_M \tag{6}$$

in one time step. The shortened length is positive, unless $P_0 = P_1 = P_2 = \cdots P_{M-1} = P_M = 0$, i.e., $S=0$. Therefore, in the asymptotic steady state of the system, L and S will no longer change, and at least one of them becomes zero. Hence, it is not possible for long and short spacings to coexist in the asymptotic steady state.

IV. ANALYTICAL SOLUTION OF ASYMPTOTIC VELOCITY

For the low-car-density case $[\rho < 1/(M+2)]$, it is apparent that in the asymptotic steady state, $L>0$ and $S=0$. Hence

$$C_n(t) \geqslant M+1, \quad \forall\, n. \tag{7}$$

In this case, stochastic delay will no longer occur, and all the cars will move forward with a maximum speed M. The average car speed of traffic flow is

$$\langle V(t\to\infty)\rangle = M. \tag{8}$$

For the high-car-density case $[\rho > 1/(M+2)]$, it is apparent that in the asymptotic steady state $S>0$ and $L=0$. Hence

$$C_n(t) \leqslant M+1, \quad \forall\, n. \tag{9}$$

The length of every intercar spacing cannot be larger than $M+1$. Therefore, the average speed of traffic flow in the asymptotic steady state is

$$\langle V(t\to\infty)\rangle = \sum_{i=1}^{M} P_i[i(1-f) + (i-1)f] + MP_{M+1}$$
$$= \sum_{i=1}^{M} P_i(i-f) + MP_{M+1} \tag{10}$$

A. $v_{\max} = M = 1$

In this case, the high density case refers to $\rho \geqslant 1/3$, and hence $P_n = 0$, $\forall\, n \geqslant 3$. This implies that only P_0, P_1, and P_2 are nonzero. To obtain the nonvanishing P_j, we introduce $N_{i\to j}$ to describe the number of intercar spacings with a change in length from i at time t to j at time $t+1$. The probability of finding an intercar spacing with length i at time t and length j at time $t+1$ is

$$W_{i\to j}(t) \equiv N_{i\to j}(t)/N. \tag{11}$$

From Eqs. (1)–(3), we can write all the nonzero $W_{i\to j}$ as

$$W_{0\to1}=P_0[(1-f)P_1+P_2],$$

$$W_{0\to2}=0,$$

$$W_{1\to0}=P_1[(1-f)P_0+f(1-f)P_1],$$

$$W_{1\to2}=P_1[f(1-f)P_1+fP_2],$$

$$W_{2\to0}=0,$$

$$W_{2\to1}=P_2(P_0+fP_1).$$

For $|i-j|\geqslant 3,\ W_{i\to j}=0$.

When the system approaches its asymptotic steady state, all the P_j is cease to change. So the following detailed balance condition for the steady state holds:

$$\sum_{i\neq m} W_{i\to m}=\sum_{i\neq m} W_{m\to i}, \quad \forall\, m. \tag{12}$$

When m is equal to 0, 1, and 2, three detailed balance equations can be written as

$$W_{0\to1}+W_{0\to2}=W_{1\to0}+W_{2\to0},$$

$$W_{1\to0}+W_{1\to2}=W_{0\to1}+W_{2\to1},$$

$$W_{2\to0}+W_{2\to1}=W_{0\to2}+W_{1\to2}.$$

Substituting the expressions for $W_{i\to j}$ into the above three equations, we obtain

$$P_0P_2=f(1-f)P_1^2. \tag{13}$$

Normalization requires that P_0, P_1, and P_2 satisfy the equations

$$\sum_i P_i=P_0+P_1+P_2=1 \tag{14}$$

and

$$\sum_i iP_i=P_1+2P_2=\overline{C}=1/\rho-1. \tag{15}$$

From Eqs. (13)–(15), we obtain a quadratic equation for P_0,

$$(2f-1)^2P_0^2+[(2f-1)^2(\overline{C}-2)+1]P_0-f(1-f)(\overline{C}-2)^2=0, \tag{16}$$

with its root given by

$$P_0=\frac{-[(2f-1)^2(\overline{C}-2)+1]+\sqrt{(2f-1)^2(\overline{C}-2)\overline{C}+1}}{2(2f-1)^2}. \tag{17}$$

Hence, the asymptotic average speed of traffic flow is

$$\langle V(t\to\infty)\rangle=\frac{\overline{C}+\dfrac{1}{2f-1}\left[\sqrt{(2f-1)^2(\overline{C}-2)\overline{C}+1}-1\right]}{2}=\frac{1}{2}\left(-1+\frac{1}{\rho}+\frac{-1+\sqrt{1+\dfrac{(2f-1)^2(\rho-1)(3\rho-1)}{\rho^2}}}{2f-1}\right). \tag{18}$$

For $f=1/2$, which is an removable singular point,

$$\langle V(t\to\infty)\rangle=\overline{C}/2=(1/\rho-1)/2. \tag{19}$$

Equations (18) and (19) give the asymptotic $\langle V(t\to\infty)\rangle$ as a function of f and ρ in the high density case with $M=1$.

B. $v_{\max}=M=2$

In this case, $\rho\geqslant 1/4$, hence $P_n=0$, $\forall\, n\geqslant 4$. This implies that only P_0, P_1, P_2, and P_3 are nonzero. From Eqs. (1)–(3), we can write the nonzero $W_{i\to j}$ as

$$W_{0\to1}=P_0[(1-f)P_1+fP_2],$$

$$W_{0\to2}=P_0[(1-f)P_2+P_3],$$

$$W_{0\to3}=0,$$

$$W_{1\to0}=P_1[(1-f)P_0+f(1-f)P_1],$$

$$W_{1\to2}=P_1\{f(1-f)P_1+[f^2+(1-f)^2]P_2+(1-f)P_3\},$$

$$W_{1\to3}=P_1[(1-f)P_2+fP_3],$$

$$W_{2\to0}=P_2[(1-f)P_0+f(1-f)P_1],$$

$$W_{2\to1}=P_2\{fP_0+[f^2+(1-f)^2]P_1+(1-f)P_2\},$$

$$W_{2\to3}=P_2[f(1-f)P_2+fP_3],$$

$$W_{3\to0}=0,$$

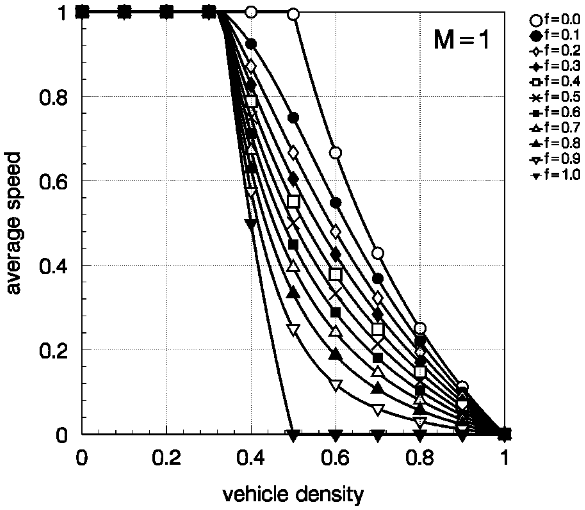

FIG. 1. The fundamental diagram with the maximum car velocity $M=1$ and for different stochastic delay probabilities *f*. The solid curves are theoretical results. The points with different symbols represent results of numerical simulations. The curves from the top down along the velocity axis correspond to different values of *f* ranging from $f=0$ to 1, in steps of 0.1.

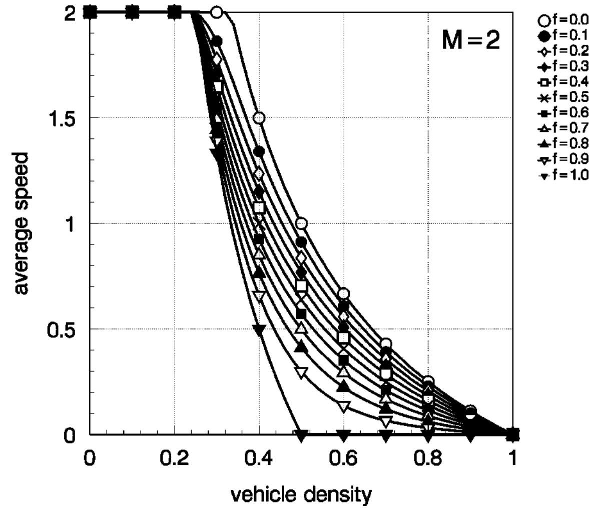

FIG. 2. The fundamental diagram with the maximum car velocity $M=2$ and for different stochastic delay probabilities *f*. The solid curves are theoretical results. The points with different symbols represent numerical simulations. The curves from the top down along the velocity axis correspond to different values of *f* ranging from $f=0$ to 1 in steps of 0.1.

$$W_{3\to1}=P_3(P_0+fP_1),$$

$$W_{3\to2}=P_3[(1-f)P_1+fP_2].$$

Substituting the above expressions into the detailed balance condition in Eq. (12), we obtain the following set of four equations:

$$fP_0P_2+P_0P_3-f(1-f)P_1^2-f(1-f)P_1P_2=0, \quad (20)$$

$$2fP_0P_2+P_0P_3-2f(1-f)P_1^2-f(1-f)P_1P_2 -(1-f)P_1P_3+f(1-f)P_2^2=0, \quad (21)$$

$$fP_0P_2-P_0P_3-f(1-f)P_1^2+f(1-f)P_1P_2-2(1-f)P_1P_3 +2f(1-f)P_2^2=0, \quad (22)$$

$$P_0P_3-f(1-f)P_1P_2+(1-f)P_1P_3-f(1-f)P_2^2=0. \quad (23)$$

Note that only two of these, e.g., Eqs. (20) and (23), are independent. Combining Eqs. (20) and (23) with the normalization conditions

$$\sum_i P_i=P_0+P_1+P_2+P_3=1 \quad (24)$$

and

$$\sum_i iP_i=P_1+2P_2+3P_3=\overline{C}=1/\rho-1, \quad (25)$$

we can solve for P_0, P_1, P_2, and P_3, and obtain the asymptotic traffic flow velocity for $M=2$.

V. DISCUSSION

In order to compare with the analytical results, we carried out numerical simulations on a 1D chain with 1000 cars. The length of the chain was adjusted so as to give the desired car density. Periodic boundary condition was imposed. The first 20000 time steps were excluded from the averaging procedure so as to remove the transient behavior. The averages were taken over the next 80000 time steps. Figures 1 and 2 show a comparison between results obtained from numerical simulations and our mean field theory for the cases of $M=1$ and 2 over the entire range of density ρ. The curves are theoretical results, while the symbols represent results of numerical simulations. The curves from the top down along the velocity axis correspond to different values of *f* ranging from 0 to 1. Excellent agreement between simulations and our theory is found.

From the fundamental diagrams of our model, it is noted that when the car density is low enough $[\rho\leqslant 1/(M+2)]$, all the intercar spacings will not be shorter than $M+1$, and all the cars will not be delayed, leading to traffic flow in its maximum velocity $(V=M)$. This situation is more realistic in that, in real traffic, no driver would tend to slow down his car when it is far away from the car ahead. In the high density case, the stochastic delay in our model represents better safety than that of the FI model, and leads to a much higher asymptotic average velocity of traffic flow than that in the NS model.

In summary, we introduced a model with stochastic delays for cars following the trail of the car ahead. Its evolution and fundamental diagram are quite different from the NS and FI models, even in the simplest case of $M=1$. We studied the evolution of the intercar spacings, and obtained its fundamental diagram by an analytical COMF approach. The results show an exact agreement between numerical simulations and our theory.

The analysis of the dynamical evolution of our model may give us a clearer physical picture of how the acceleration and stochastic delay rules affect the evolution and the corresponding asymptotic steady state. It will also provide us with better ideas on developing analytical approaches to other traffic flow CA models such as the NS model, for which no exact analytical approach has been established.

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